

The bifurcation locus for numbers of bounded type

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Abstract

We define a family $\mathcal{B}(t)$ of compact subsets of the unit interval which generalizes the sets of numbers whose continued fraction expansion has bounded digits. We study how the set $\mathcal{B}(t)$ changes as one moves the parameter t , and see that the family undergoes period-doubling bifurcations and displays the same transition pattern from periodic to chaotic behaviour as the usual family of quadratic polynomials. The set \mathcal{E} of bifurcation parameters is a fractal set of measure zero. We also show that the Hausdorff dimension of $\mathcal{B}(t)$ varies continuously with the parameter, and the dimension of each individual set equals the dimension of the corresponding section of the bifurcation set \mathcal{E} .

1 Introduction

Every $x \in [0, 1] \setminus \mathbb{Q}$ can be encoded in a unique way by its continued fraction expansion

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ddots}}} = [0; a_1, a_2, a_3, \dots],$$

where $a_i \in \mathbb{N}_+$ are called the *partial quotients* of x . We say that x is a *number of bounded type* if its partial quotients are bounded.

More precisely, given $N \in \mathbb{N}$ one can consider the set of numbers whose partial quotients are bounded by N :

$$\mathcal{B}_N := \{x = [0; a_1, \dots, a_k, \dots] : 1 \leq a_k \leq N \quad \forall k \geq 1\} \quad (1)$$

The sets \mathcal{B}_N have been studied by several authors (see [He2] and the references therein for an account) and their Hausdorff dimensions have been computed very precisely [JP]. Moreover, numbers of bounded type play a key role, not only in number theory, but also in various contexts in dynamics, for instance in the theory of linearization of analytic maps, and KAM theory.

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Let us recall that the Gauss map $G(x) := \{\frac{1}{x}\}$ acts as a shift operator on the continued fraction expansion, hence notice that \mathcal{B}_N can be described in terms of the orbit for the Gauss map as

$$\mathcal{B}_N = \{x \in [0, 1] : G^k(x) \geq \frac{1}{N+1} \quad \forall k \in \mathbb{N}\}$$

The sets \mathcal{B}_N give a filtration of the set of numbers of bounded type $\mathcal{B} := \cup_{N \geq 1} \mathcal{B}_N$. More generally, if we consider sets of points whose orbit stays above a certain fixed threshold t , we obtain a filtration of \mathcal{B} with a continuous parameter which refines \mathcal{B}_N :

Definition 1. Given $t \in [0, 1]$,

$$\mathcal{B}(t) := \{x \in [0, 1] : G^k(x) \geq t \quad \forall k \in \mathbb{N}\} \quad (2)$$

Clearly, $\mathcal{B}_N = \mathcal{B}\left(\frac{1}{N+1}\right)$, and $\mathcal{B}(t) \subseteq \mathcal{B}(t')$ if $t \geq t'$. The union of all $\mathcal{B}(t)$ is the set of all numbers of bounded type $\mathcal{B} = \cup_{t>0} \mathcal{B}(t)$. Every $\mathcal{B}(t)$ is a compact subset of the unit interval, and it is nowhere dense of measure zero whenever $t > 0$.

Examples:

1. Let $g := [0; \overline{1}] = \frac{\sqrt{5}-1}{2}$, it is easy to see that $\mathcal{B}(g) = \{g\}$, in fact $\mathcal{B}(t) = \{g\}$ for all t between $[0; \overline{2}]$ and g , while $\mathcal{B}(t) = \emptyset$ for all $t > g$;
2. for $\alpha = [0; \overline{2, 1}]$, $\mathcal{B}(\alpha) = \mathcal{B}_2 = \mathcal{B}(1/3)$ is the set of numbers whose continued fraction contains only the digits 1 and 2;
3. for $t = g^2 = [0; 2, \overline{1}]$ we get that $\mathcal{B}(g^2)$ is the set of numbers whose continued fraction contains only the digits 1 and 2 and such that between any two 2s there is an even number of 1s. In formulas, $\mathcal{B}(g^2) = A_0 \cup \phi A_0$ with $A_0 := \{x = [0; X], \quad X \in \{(2), (1, 1)\}^{\mathbb{N}}\}$ and $\phi(x) := 1/(1+x)$.

We will study how the set $\mathcal{B}(t)$ changes as one moves the parameter t . We will see that our family of sets undergoes period-doubling bifurcations, and it displays the same transition pattern from periodic to chaotic behaviour as the usual family of quadratic polynomials.

Let us define the *bifurcation locus* of a function of the unit interval as the complement of the set of points where the function is locally constant: the first result (section 2.2) is the

Proposition 1. *The bifurcation locus for the set function $t \mapsto \mathcal{B}(t)$ is exactly the set*

$$\mathcal{E} := \{x \in [0, 1] : G^k(x) \geq x \quad \forall k \in \mathbb{N}\}$$

The bifurcation set \mathcal{E} appears in several, seemingly unrelated contexts: it was introduced in [CT] to describe the set of phase transitions of the entropy of α -continued fraction transformations, and it is essentially the same object as the spectrum of recurrence quotients for cutting sequences of geodesics on the torus studied by Cassaigne ([Ca], Theorem 1.1).

The set \mathcal{E} is also related to kneading sequences of unimodal maps: indeed, in [BCIT] it is explained how one can construct an explicit dictionary between

\mathcal{E} and the set of external rays landing on the real section of the boundary of the Mandelbrot set \mathcal{M} . In this sense, this paper complements the one just mentioned, since our sets $\mathcal{B}(t)$ are the formal analogue to the family of Julia sets for real quadratic polynomials $p_c(z) = z^2 + c$. More precisely, under the same dictionary as in [BCIT], one can pass from each $t \in \mathcal{E}$ to a definite point $c \in \partial\mathcal{M} \cap \mathbb{R}$, in such a way that $\mathcal{B}(t)$ corresponds to the set of external rays landing on the *Hubbard tree* of the corresponding real quadratic polynomial p_c .

The sets $\mathcal{B}_N = \mathcal{B}(1/(N+1))$ are Cantor sets for all $N \geq 2$, but this is not always the case in the family $\mathcal{B}(t)$, and indeed the topology changes wildly with the parameter. The goal of section 3 is to describe each $\mathcal{B}(t)$ by determining the connected components of its complement.

As a consequence, we will characterize the set of values such that $\mathcal{B}(t)$ contains isolated points, and prove that these special values are exactly the points of discontinuity for the function $t \mapsto \text{H.dim } \mathcal{B}(t)$, where we consider the Hausdorff topology on compact sets (see section 3.2).

Another interesting way to study the variation in structure of the family $\mathcal{B}(t)$ consists in considering the Hausdorff dimension $\text{H.dim } \mathcal{B}(t)$ as a function of the parameter t (see figure 1 for a picture). Such function is locally constant almost everywhere, and displays a devil's staircase behaviour. Nonetheless, we can prove the

Theorem 1. *The function*

$$t \mapsto \text{H.dim } \mathcal{B}(t)$$

is continuous.

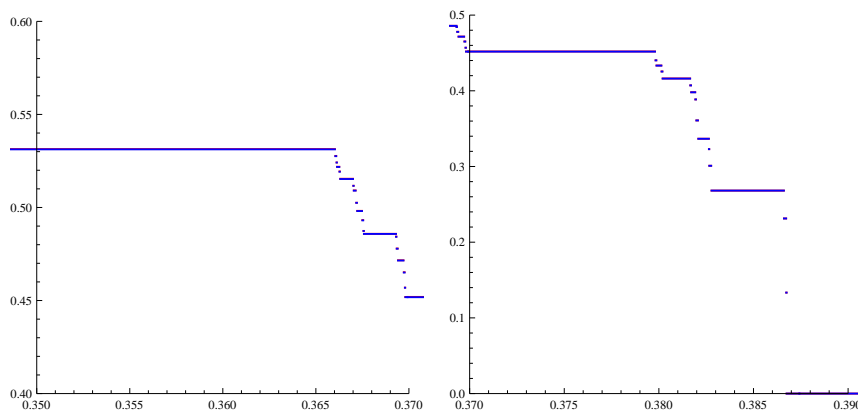


Figure 1: Hausdorff dimension of $\mathcal{B}(t)$ for $0.35 \leq t \leq 0.37$ (left) and for $0.37 \leq t \leq 0.39$ (right). The dimension drops to 0 around the “Feigenbaum value” $c_F = 0.3867499707\dots$ (see section 3.1).

Pushing further the comparison with Julia sets, let us recall that also in the quadratic polynomial case one can consider the dependence of Hausdorff dimension of Julia sets on the parameter, or the dependence of topological entropy of unimodal maps on the kneading parameter. In the latter case, which

turns out to be much more similar to ours, entropy varies continuously, and its plateaux have been characterized in [Do2].

Moreover, we will relate the Hausdorff dimension of each $\mathcal{B}(t)$ to the dimension of a section of parameter space:

Theorem 2. *For every $t \in [0, 1]$, let $\mathcal{E}(t) := \mathcal{E} \cap [t, 1]$. Then*

$$\text{H.dim } \mathcal{B}(t) = \text{H.dim } \mathcal{E}(t)$$

Theorem 1 is proved using an argument of Moreira, which was used to study the dimension of sections of the Markoff spectrum [Mo]. Since the argument is quite technical, it will be postponed to section 6.1.

The proof of theorem 2, instead, is given in section 5 and relies on renormalization techniques which we develop in section 4 and, as far as we know, are new to the subject of continued fractions.

Namely, given a rational number r , we will define a *tuning map* τ_r which maps parameter space into a copy of itself; this tuning is a natural translation, via the dictionary of [BCIT], of the classical Douady-Hubbard tuning for quadratic polynomials ([Do1], [DH]). Each tuning map τ_r maps the unit interval into a *tuning window* W_r ; the set UT of *untuned parameters* is the complement of the union of all tuning windows¹. The proof consists of three steps:

1. Since the functions $x \mapsto \mathcal{B}(x)$ and $x \mapsto \mathcal{E}(x)$ are both locally constant on the complement on \mathcal{E} , it is enough to prove

$$\text{H.dim } \mathcal{B}(x) = \text{H.dim } \mathcal{E}(x) \quad \text{for all } x \in \mathcal{E}$$

2. We first consider the case (section 4.2) when x is an untuned parameter: in that case, one proves that there is almost an embedding of $\mathcal{E}(x)$ in $\mathcal{B}(x)$, or more precisely

Proposition 2. *Given any point $x \in UT$ and any $y > x$, then $\mathcal{E}(x)$ contains a Lipschitz image of $\mathcal{B}(y)$.*

and equality of dimensions follows from continuity (theorem 1).

3. We proceed by induction, using renormalization. Indeed, we prove formulas which relate the dimensions of $\mathcal{E}(x)$ and $\mathcal{B}(x)$ to the dimensions of their tuned copies, and using the untuned case as the base step we prove equality of dimensions for all parameters which are finitely-renormalizable. Such parameters are dense in \mathcal{E} , hence the general claim again follows from continuity.

The renormalization approach is also useful to study the modulus of continuity of the function $t \mapsto \text{H.dim } \mathcal{B}(t)$, which turns out not to be Hölder-continuous at the fixed point c_F of the tuning operator $\tau_{1/2}$ (see section 4.1 and figure 2).

Theorem 2 is formally similar to a widely known result of Urbański [Ur] for smooth circle maps. Our methods, however, are completely different, since we do not use any thermodynamic formalism and we introduce the renormalization scheme. Moreover, the geometry of the bifurcation loci is different, since in

¹Since tuning is the inverse of renormalization, untuned parameters can also be called *non-renormalizable*.

Urbański's case the bifurcation locus for the set function is a Cantor set and it coincides with the bifurcation locus for the dimension, while neither statement is true in our case because of isolated points and tuning windows.

Theorem 2 also provides an answer to a question raised in [Ca] (see section 5.1):

Proposition 3. *Let \mathcal{R} be the recurrence spectrum for Sturmian sequences. Then, for each positive integer N , the Hausdorff dimension of $\mathcal{R} \cap [N+2, N+3]$ equals the Hausdorff dimension of \mathcal{B}_N .*

The computation of Hausdorff dimension for the sections of the recurrence spectrum is thus reduced to estimating the dimension of \mathcal{B}_N . Since the latter is a regular Cantor set, there are very efficient algorithms to compute its dimension ([JP], [He1]).

2 Preliminaries

2.1 Notation

The continued fraction expansion of a number x will be denoted by $x = [0; a_1, a_2, \dots]$, or by $x = [0; S]$ where $S = (a_1, a_2, \dots)$ is a (finite or infinite) string of positive integers. Let us note that irrational numbers have infinite continued fraction expansions, and every rational number $r \in \mathbb{Q} \cap (0, 1)$ has exactly two finite expansions, one of even length and one of odd length; we will usually denote by S_0 the string of even length, and by S_1 the string of odd length ($r = [0; S_0] = [0; S_1]$). Moreover, the expression $x = [0; \overline{S}]$ will denote the quadratic irrational whose continued fraction expansion is periodic, given by S repeated infinitely many times. The length of the string S will be denoted by $|S|$.

If S is a finite string of positive integers we will denote by $q(S)$ the denominator of the rational number given whose c.f. expansion is S , i.e. $\frac{p(S)}{q(S)} = [0; S]$ with $(p(S), q(S)) = 1$, $q(S) > 0$.

Let us also recall the well known estimate

$$q(S)q(T) \leq q(ST) \leq 2q(S)q(T). \quad (3)$$

Moreover, we can define an action of the semigroup of finite strings (with the operation of concatenation) on the unit interval. Indeed, for each S , $S \cdot x$ is the number obtained by appending the string S at the beginning of the continued fraction expansion of x . We shall also use the notation $f_S(x) := S \cdot x$. If $S = (a_1, \dots, a_n)$ we can also write, by identifying matrices with Möbius transformations,

$$S \cdot x := \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix} \dots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} \cdot x$$

It is easy to check that concatenation of strings corresponds to composition $(ST) \cdot x = S \cdot (T \cdot x)$; moreover, the map f_S is increasing if $|S|$ is even, decreasing if it is odd. The image of f_S is a *cylinder set*

$$I(S) := \{x = S \cdot y, y \in [0, 1]\}$$

which is a closed interval with endpoints $[0; a_1, \dots, a_n]$ and $[0; a_1, \dots, a_n + 1]$. The map f_S is a contraction of the unit interval, and it is easy to see that

$$\frac{1}{2q(S)^2} \leq |f'_S(x)| \leq \frac{1}{q(S)^2} \quad \forall x \in [0, 1] \quad (4)$$

hence the length of $I(S)$ is bounded by

$$\frac{1}{2q(S)^2} \leq |I(S)| \leq \frac{1}{q(S)^2} \quad (5)$$

Given two strings of positive integers $S = (a_1, \dots, a_n)$ and $T = (b_1, \dots, b_n)$ of equal length, let us define the *alternate lexicographic order* as

$$S < T \text{ if } \exists k \leq n \text{ s.t. } a_i = b_i \ \forall 1 \leq i \leq k-1 \text{ and } \begin{cases} a_n < b_n & \text{if } n \text{ even} \\ a_n > b_n & \text{if } n \text{ odd} \end{cases}$$

The importance of such order lies in the fact that (given two strings of equal length) $S < T$ iff $[0; S] < [0; T]$. Moreover, in order to compare quadratic irrationals with periodic expansion, it is useful the following *string lemma* ([CT], lemma 2.12): for any pair of strings S, T of positive integers,

$$ST < TS \Leftrightarrow [0; \overline{S}] < [0; \overline{T}] \quad (6)$$

The order $<$ is a total order on the strings of positive integers of fixed given length; to be able to compare strings of different lengths we will define the partial order

$$S << T \text{ if } \exists i \leq \min\{|S|, |T|\} \text{ s.t. } S_1^i < T_1^i$$

where $S_1^i = (a_1, \dots, a_i)$ denotes the truncation of S to the first i characters. Let us note that:

1. If $|S| = |T|$, then $S < T$ iff $S << T$.
2. If S, T, U are any strings, $S << T \Rightarrow SU << T, S << TU$
3. If $S << T$, then $S \cdot z < T \cdot w$ for any $z, w \in (0, 1)$.

In general, the computation of the Hausdorff dimension of a compact subset of the interval is a non-trivial task: however, there is a family of sets whose dimension we can estimate easily, namely sets of numbers whose continued fraction expansion is given by concatenations of words of some finite alphabet:

Definition 2. *Given a finite set of strings \mathcal{A} , the regular Cantor set defined by \mathcal{A} is the set*

$$K(\mathcal{A}) := \{x = [0; W_1, W_2, \dots] : W_i \in \mathcal{A} \ \forall i \geq 1\}$$

Indeed, if the alphabet \mathcal{A} is not redundant, in the sense that for each W_i and W_j distinct elements of \mathcal{A} , the cylinders $I(W_i)$ and $I(W_j)$ have disjoint interior, then a regular Cantor set is a particular case of an *iterated function system* and its Hausdorff dimension can be estimated in a standard way ([Fa], chapter 9). To get a rough estimate, just let $m_1 := \inf_{x \in [0, 1]} \inf_{W \in \mathcal{A}} |f'_W|$ and

$m_2 := \sup_{x \in [0,1]} \sup_{W \in \mathcal{A}} |f'_W|$ be respectively the smallest and largest contraction factors of the maps f_W (notice $m_2 \leq 1$), and let N be the cardinality of \mathcal{A} . Then

$$\frac{\log N}{-\log m_1} \leq \text{H. dim } K(\mathcal{A}) \leq \frac{\log N}{-\log m_2} \quad (7)$$

2.2 Generalized bounded type numbers and their bifurcation locus

Let us start by proving some elementary facts, which will yield proposition 1 stated in the introduction, namely that the bifurcation locus of the family $\mathcal{B}(t)$ is the *exceptional set*

$$\mathcal{E} := \{x \in [0, 1] \mid G^k(x) \geq x \ \forall k \in \mathbb{N}\}$$

Lemma 1. *The sets $\mathcal{B}(t)$ have the following properties:*

- (i) $\mathcal{B}(0) = [0, 1]$; $\mathcal{B}(t) = \emptyset$ if $t > g = \frac{\sqrt{5}-1}{2}$, in fact $t \mapsto \mathcal{B}(t)$ is monotone decreasing;
- (ii) $\mathcal{B}(t)$ is forward-invariant for the Gauss map G ;
- (iii) $\mathcal{B}(t)$ is closed and, if $t > 0$, with no interior and of zero Lebesgue measure;
- (iv) the union $\bigcup_{t>0} \mathcal{B}(t)$ is the set of bounded type numbers;
- (v) $\bigcap_{t'<t} \mathcal{B}(t') = \mathcal{B}(t)$;
- (vi) $\mathcal{E} = \{t \in [0, 1] : t \in \mathcal{B}(t)\}$

Proof. Points (i), (ii), (iv), (v), (vi) are immediate by definition.

(iii) Let us consider the Farey map $F : [0, 1] \rightarrow [0, 1]$

$$F(x) := \begin{cases} \frac{x}{1-x} & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{1-x}{x} & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

One can easily check that if $x := [0; a_1, a_2, a_3, \dots]$ then $F(x) = [0; a_1 - 1, a_2, a_3, \dots]$ if $a_1 > 1$ (while $F(x) = [0; a_2, a_3, \dots]$ in the case $x := [0; 1, a_2, a_3, \dots]$) and so it is clear that for each $x \in [0, 1]$

$$\inf_{k \geq 1} G^k(x) = \inf_{k \geq 1} F^k(x).$$

Therefore one can write

$$\mathcal{B}(t) = \{x \in [0, 1] : F^k(x) \geq t \ \forall k \in \mathbb{N}\}$$

which is closed by continuity of F . For $t > 0$, $\mathcal{B}(t)$ has no interior because it does not contain any rational number, and it has measure zero by ergodicity of the Gauss map. \square

Since $\mathcal{B}(t)$ is a non-empty compact set for $0 \leq t \leq g$, for such values of t one can define the function

$$\ell(t) := \min \mathcal{B}(t)$$

We shall list some elementary properties of ℓ .

Lemma 2. *The function $t \mapsto \ell(t)$ is monotone increasing and*

- (i) *For any $0 \leq t \leq g$, $\ell(t) \in \mathcal{E}$*
- (ii) *$t \leq \ell(t) \forall t \in [0, g]$*
- (iii) *$t = \ell(t) \iff t \in \mathcal{E}$*
- (iv) *ℓ is left-continuous: $\ell(t) = \sup_{t' < t} \ell(t') = \lim_{t' \rightarrow t^-} \ell(t')$*
- (v) *if (α, β) is a connected component of $[0, g] \setminus \mathcal{E}$ then*

$$\begin{aligned} \ell(t) &= \beta \\ \mathcal{B}(t) &= \mathcal{B}(\beta) \end{aligned} \quad \forall t \in (\alpha, \beta]$$

Proof. (i): since $\ell(t) \in \mathcal{B}(t)$, $G^n(\ell(t)) \geq \ell(t)$, hence $\ell(t) \in \mathcal{E}$. (ii): $x \in \mathcal{B}(t) \Rightarrow x \geq t$, hence $\ell(t) \geq t$. (iii) is a consequence of lemma 1-(vi):

$$t = \ell(t) \iff t \in \mathcal{B}(t) \iff t \in \mathcal{E}.$$

(iv) follows from lemma 1-(v). (v): let us pick t such that $\alpha < t < \beta$. Since (α, β) is a connected component of $[0, g] \setminus \mathcal{E}$ we have $\beta \in \mathcal{E}$ and so $\beta \in \mathcal{B}(\beta) \subset \mathcal{B}(t)$, and

$$\beta \geq \min \mathcal{B}(t) = \ell(t).$$

On the other hand, since $(\alpha, \beta) \cap \mathcal{E} = \emptyset$ and $\ell(t) \in \mathcal{E} \cap [t, 1]$ it follows that

$$\ell(t) \geq \beta$$

We have thus proved that $\ell(t) = \beta$. Now, from (ii) and monotonicity, $\mathcal{B}(\ell(t)) \subseteq \mathcal{B}(t)$. Moreover, if $x \in \mathcal{B}(t)$, by G -invariance $G^n(x) \in \mathcal{B}(t)$, hence $G^n(x) \geq \ell(t)$ and $x \in \mathcal{B}(\ell(t))$, hence $\mathcal{B}(t) = \mathcal{B}(\ell(t)) = \mathcal{B}(\beta)$. \square

From lemma 2 it also follows

$$\ell(t) = \min(\mathcal{E} \cap [t, 1])$$

Proof of proposition 1. By lemma 2-(v), the function $t \mapsto \mathcal{B}(t)$ is locally constant outside \mathcal{E} . On the other hand, if $t \in \mathcal{E}$, then $t \in \mathcal{B}(t)$ by definition, but $t \notin \mathcal{B}(t')$ for any $t' > t$, so t must belong to the bifurcation set. \square

3 A complementary point of view

If one wants to figure out the structure of a closed, nowhere dense set $C \subset [0, 1]$ (even just in order to plot a picture), one usually tries to describe its complement: indeed, while C is typically uncountable and totally disconnected, its complement $A := [0, 1] \setminus C$ is a countable union of open connected components (intervals):

$$A = \bigcup_{j \in \mathbb{N}} I_j.$$

We recall the following simple fact ([CT], lemma 2.1):

If $(a, b) \subset [0, 1]$ there is a **unique** rational number $p/q \in (a, b) \cap \mathbb{Q}$ with minimal denominator, i.e. such that for all other $p'/q' \in (a, b) \cap \mathbb{Q}$, $q \leq q'$. We shall call such a "minimal" rational the pseudocenter of the interval (a, b) .

The above fact provides a canonical way of labelling the connected components of $[0, 1] \setminus C$; in fact if C is a closed, possibly fractal set of irrational numbers which is defined imposing a certain condition on the continued fraction of its elements, it often happens that the pseudocenters of the connected components of $[0, 1] \setminus C$ are rational values which satisfy, at a finite stage, the same property which defines C : this is known to be true for diophantine-type conditions arising in the solution of cohomological equations [MS], for the sublevels of the Brjuno function [CM] and -indeed- also for \mathcal{E} and $\mathcal{B}(t)$, as we shall see in a moment.

3.1 Extremal rational numbers and the exceptional set

The bifurcation set \mathcal{E} is a totally disconnected, nowhere dense compact set. An explicit description of the connected components of its complement was given in [CT]. In order to explain the construction, let us consider the following class of strings:

Definition 3. A finite string of positive integers S is called extremal if

$$XY < YX$$

for every splitting $S = XY$ where X, Y are nonempty strings.

Such strings will generate a particular class of rational numbers:

Definition 4. Let $r \in \mathbb{Q} \cap (0, 1)$ be a rational number, and $r = [0; S_1]$ its continued fraction expansion of odd length. Then r is called extremal if S_1 is extremal.

We now have the tools to describe the complement of \mathcal{E} . Indeed, given $r = [0; S_0] = [0; S_1]$ an extremal rational number, $0 < r < 1$, and let $I_r := (\alpha^-, \alpha^+)$ where $\alpha^- = [0; \overline{S_1}]$, $\alpha^+ = [0; \overline{S_0}]$. Moreover, let $I_1 := (g, 1]$, and $\mathbb{Q}_E = \{r \in \mathbb{Q} \cap (0, 1), r \text{ extremal}\} \cup \{1\}$. Then

$$[0, 1] \setminus \mathcal{E} := \bigcup_{r \in \mathbb{Q}_E} I_r \quad (8)$$

Notice that r is the pseudocenter of I_r , hence the set of pseudocenters of all connected components of the complement of \mathcal{E} is just the set \mathbb{Q}_E of rational extremal numbers.

It is also possible to describe the topology of \mathcal{E} in detail, and more precisely identify its isolated points. Indeed, if $r = [0; S_1]$ is extremal, one can check that $r_1 = [0; S_1 S_0]$ is also extremal, and $\mathcal{E} \cap [r_1, r]$ consists of the single, isolated point $\alpha_1 = [0; \overline{S_1}]$. All isolated points are obtained in this way, and moreover they are grouped in *period-doubling cascades*: indeed, one can reiterate the above process starting with r_1 and produce a new rational number $r_2 := [0; S_1 S_0 S_1 S_1]$ and the isolated point $\alpha_2 = [0; \overline{S_1 S_0 S_1 S_1}]$ of \mathcal{E} . Such a sequence of isolated points accumulates on a point $\alpha_\infty \in \mathcal{E}$, which has not a periodic continued fraction expansion.

For instance, by starting with $r = \frac{1}{2} = [0; 2] = [0; 1, 1]$, one gets the family

$$[0; \overline{2}], [0; \overline{2, 1, 1}], [0; \overline{2, 1, 1, 2, 2}], [0; \overline{2, 1, 1, 2, 2, 2, 1, 1, 2, 1, 1}]$$

which tends to the point $c_F = 0.3867499707\dots$. Notice that in the dictionary of [BCIT], this point corresponds to the Feigenbaum parameter for unimodal maps (hence the notation c_F). We will see in section 4 how these cascades are produced by iteration of tuning operators.

Let us also notice that if r is extremal and its continued fraction expansion of even length is $r = [0; S_0]$, then for all splittings $S_0 = XY$

$$XY \leq YX$$

3.2 Anatomy of generalized bounded type numbers

In this section, we will provide an analogue description for the set $\mathcal{B}(t)$, by producing the set of pseudocenters of all connected components of its complement. We need two definitions:

Definition 5. Let $t \in]0, g]$ be a fixed value, $r := [0; S^\pm] \in \mathbb{Q} \cap (0, 1)$ and let us define

$$V_r := (S^- \cdot t, S^+ \cdot t) \quad (9)$$

We also define $V_0 = [0, t[$ and $V_1 :=]1/(1+t), 1]$. We shall call V_r the t -gap generated by $r \in \mathbb{Q} \cap [0, 1]$.

Definition 6. Let $t \in (0, 1)$ and $r \in \mathbb{Q} \cap [0, 1]$. We say that $m \in \mathbb{N}$ is the running time of r if

$$G^m(r) = 0 \quad \text{but } G^k(r) > 0 \quad \forall 0 \leq k \leq m-1 \quad (10)$$

We say that r is a t -label if the G -orbit of r never enters the interval $(0, \ell(t))$; this means that if m is the running time of r

$$G^m(r) = 0 \quad \text{but } G^k(r) > \ell(t) \quad \forall 0 \leq k \leq m-1 \quad (11)$$

We shall denote by \mathbb{Q}_t the set of all t -labels.

We will prove that t -gaps provide a complete description of $\mathcal{B}(t)$ by determining the connected components of its complement:

Proposition 4. Let $t \in [0, g]$, and $\beta := \ell(t) \in \mathcal{E}$. Then the connected components of the complement of $\mathcal{B}(t)$ are precisely the β -gaps generated by the t -labels, hence

$$\mathcal{B}(t) = [0, 1] \setminus \bigcup_{r \in \mathbb{Q}_t} V_r$$

Corollary 1. $\mathcal{B}(t)$ contains isolated points if and only if $\beta := \ell(t)$ is isolated in \mathcal{E} .

Proof. Let us set $\beta := \ell(t)$. Let us recall that β is isolated in \mathcal{E} iff $\beta = [0; \overline{S}]$ with minimal period S of odd length.

Now, we shall prove that if β is isolated in \mathcal{E} , then it is isolated in $\mathcal{B}(t) = \mathcal{B}(\beta)$. Indeed, if $\beta = [0; \overline{S}]$ has odd minimal period, then $S \cdot 0 > \beta$, moreover

if $x \in (\beta, S \cdot 0)$ then $x = S \cdot y > S \cdot \beta = \beta$ for some y , hence, since $|S|$ is odd, $G^{|S|}(x) = y < \beta$. This means that x does not belong to $\mathcal{B}(t) = \mathcal{B}(\beta)$, hence β is isolated in $\mathcal{B}(\beta)$.

Conversely, if x is an isolated point of $\mathcal{B}(t)$ then $x = S \cdot \beta = T \cdot \beta$ where S, T are two strings, one of even and one of odd length; if -say- S is the longest then $S = TR$ with $|R|$ odd, therefore $R \cdot \beta = \beta$, which means that β is periodic with minimal period of odd length.

□

Corollary 2. *The points of discontinuity of the function*

$$t \mapsto \mathcal{B}(t)$$

in the Hausdorff topology are precisely the isolated points of \mathcal{E} (corresponding to period-doubling bifurcation parameters).

Proof. If A is a set, we shall set $N_\epsilon(A) := \{x : d(x, A) < \epsilon\}$; let us also recall that if A, B are two nonempty closed sets, then the Hausdorff distance $d_H(A, B)$ is smaller than ϵ iff $B \subset N_\epsilon(A)$ and $A \subset N_\epsilon(B)$; let us point out that, since the sets we are interested in are monotone, we will always have to check just one inclusion.

It is easy to prove that $t \mapsto \mathcal{B}(t)$ is everywhere left-continuous with respect to Hausdorff distance: this amounts to check that, if $t \in (0, 1)$ and $t_n \nearrow t$, then for every fixed $\epsilon > 0$ there exists \bar{n} such that $\mathcal{B}(t_n) \subset N_\epsilon(\mathcal{B}(t))$, this is a straightforward consequence of $\mathcal{B}(t) = \cap_{t' < t} \mathcal{B}(t')$.

On the other hand, because of corollary 1 the map cannot be right-continuous at isolated point of \mathcal{E} , so we just have to prove that it is right-continuous at any β a non-isolated point of \mathcal{E} ; such points are always accumulated on the right by points of \mathcal{E} , hence

$$\exists(\beta_n) \subset \mathcal{E} : \beta_n \searrow \beta \text{ as } n \rightarrow +\infty.$$

To prove right-continuity at any such $\beta \in \mathcal{E}$ it will be enough to check that for all $\epsilon > 0$ there exists \bar{n} such that $\mathcal{B}(\beta) \subset N_\epsilon(\mathcal{B}(\beta_{\bar{n}}))$. Since the set $\{S \cdot \beta : [0; S] \in \mathbb{Q}_\beta\}$ is dense in $\mathcal{B}(\beta)$ we can find a finite subset $\mathcal{S} \subset \{S : [0; S] \in \mathbb{Q}_\beta\}$ such that $\mathcal{B}(\beta) \subset N_{\epsilon/2}(E)$ with $E := \{S \cdot \beta : \beta \in \mathcal{S}\}$. Therefore if we prove that

$$E \subset N_{\epsilon/2}(\mathcal{B}(\beta_{\bar{n}})) \quad (*)$$

we are done. For $S \in \mathcal{S}$ we denote $\underline{r}(S)$ the minimum positive element of the orbit $G^k(S \cdot 0)$; since $\mathcal{S} \subset \mathbb{Q}_\beta$ we have that $\underline{r}(S) > \beta$, therefore $\underline{r} := \min_{S \in \mathcal{S}} \underline{r}(S) > \beta$ as well. Now let us choose \bar{n} such that

$$(a) \beta_{\bar{n}} \in (\beta, \underline{r}), \quad (b) |\beta_{\bar{n}} - \beta| < \epsilon/2;$$

by (a) we have that $\mathcal{S} \subset \mathbb{Q}_{\beta_{\bar{n}}}$ and $S \cdot \beta_{\bar{n}} \in \mathcal{B}(\beta_{\bar{n}})$ for all $S \in \mathcal{S}$, on the other hand (b) implies that

$$|S \cdot \beta - S \cdot \beta_{\bar{n}}| \leq |\beta - \beta_{\bar{n}}| < \epsilon/2 \quad \forall S \in \mathcal{S},$$

and so condition (*) above holds.

□

In order to prove the proposition, let us first observe some basic properties of t -labels:

- If V_r is a t -gap, then $V_r \cap \mathcal{B}(t) = \emptyset$;
- if V_r is the t -gap generated by r and $G^k(r) \neq 0$, then $G^k(V_r)$ is the t -gap generated by $G^k(r)$.
- $[0, 1] \setminus \mathcal{B}(t) = \bigcup_{r \in [0, 1] \cap \mathbb{Q}} V_r$.

In the formula above, in the union lots of overlappings occur; nonetheless we shall see that $\mathcal{B}(t)$ can be described as the disjoint union of a subfamily of β -gaps, with $\beta = \ell(t) \in \mathcal{E}$.

Lemma 3. *Let $\beta \in \mathcal{E}$ be a fixed value, $r := [0; S] \in \mathbb{Q} \cap [0, 1]$, $r > \beta$. Then $S \cdot \beta \geq \beta$.*

Proof. If $|S|$ is even the claim is trivial:

$$S \cdot \beta \geq S \cdot 0 = r > \beta.$$

If $|S|$ is odd, set $\alpha^- := [0; \overline{S}]$; α^- is the left endpoint of the quadratic interval I_r generated by r , so $\alpha^- < r$ but also $\beta \leq \alpha^-$ (because no point of \mathcal{E} can belong to I_r), hence $S\beta \geq S\alpha^- = \alpha^- \geq \beta$. \square

Lemma 4. *Let $t \in [0, 1]$, and $r := [0; S^\pm] \in \mathbb{Q}_\beta$ be a t -label, $\beta := \ell(t)$. Then the β -gap generated by r , namely the open interval $V_r := (S^- \cdot \beta, S^+ \cdot \beta)$, is a connected component of $[0, 1] \setminus \mathcal{B}(t)$.*

Proof. If $r = 0$ or $r = 1$ there is no problem. Otherwise $r = [0; S^\pm]$ and $V_r \subset [0, 1] \setminus \mathcal{B}(t)$; to prove our claim it is sufficient to check that $\partial V_r \subseteq \mathcal{B}(t)$, i.e. $\beta^\pm := S^\pm \cdot \beta \in \mathcal{B}(\beta)$. By assumption we have that $G^k(r) > \beta$ for $0 \leq k \leq m-1$; moreover $\partial V_{G^k(r)} = \{G^k(\beta^\pm)\}$ and hence $G^k(\beta^\pm) \geq \beta$ for $0 \leq k \leq m-1$ (by lemma 3) On the other hand $G^m(\beta^\pm) = \beta$ and hence $\beta^\pm \in \mathcal{B}(\beta)$. \square

Lemma 5. *Let $t \in (0, 1)$, let $V := (\xi^-, \xi^+)$ be a connected component of $[0, 1] \setminus \mathcal{B}(t)$, and let $r \in V \cap \mathbb{Q}$. The following conditions are equivalent:*

- (i) r is the pseudocenter of V ;
- (ii) r is a t -label.

(i) \Rightarrow (ii). We argue by contradiction. Assume that $r \in \mathbb{Q}$ is the pseudocenter of V but exists k_0 such that $r_1 := G^{k_0}(r) \in (0, t)$. In this case $r = [0; S_0 S_1]$ and $r_1 = [0; S_1]$; moreover setting $r_0 := [0; S_0]$ we see that the interval $[0, r_1] \subset [0, t]$, hence $S_0 \cdot [0, r_1] \subset [0, 1] \setminus \mathcal{B}(t)$, and so $S_0 \cdot [0, r_1]$ must also be contained in V , by connectedness. This is a contradiction: $r_0 = S_0 \cdot 0 \in V$ and $\text{den}(r_0) < \text{den}(r)$ imply that r cannot be the pseudocenter of V .

[(ii) \Rightarrow (i)] Let $\beta := \ell(t) \in \mathcal{E}$, so that $\mathcal{B}(t) = \mathcal{B}(\beta)$. By lemma 4, if $r := [0; S^\pm] \in V$ satisfies (ii) then $V = (S^- \cdot \beta, S^+ \cdot \beta)$ and hence r is the pseudocenter of V . \square

Proof of proposition 4. If r is a t -label, then the β -gap V_r is a connected component of $[0, 1] \setminus \mathcal{B}(t)$ by lemma 4. On the other hand, if V is a connected component, then its pseudocenter r is a t -label by lemma 5, hence $V = V_r$ by lemma 4. \square

4 Tuning

The goal of this section is to define renormalization operators for numbers of bounded type, and identify their action on parameter space.

Let $r \in \mathbb{Q} \cap (0, 1)$, and let $r = [0; S_0] = [0; S_1]$ respectively the even and odd continued fraction expansion of r . The *tuning window* associated to r is the interval $W_r := [\omega, \alpha]$ with endpoints

$$\omega := [0; S_1 \overline{S_0}]$$

$$\alpha := [0; \overline{S_0}]$$

For instance, if $r = \frac{1}{3} = [0; 3] = [0; 2, 1]$, then $\omega = [0; 3, \overline{2, 1}] = \frac{5-\sqrt{3}}{11}$, $\alpha = [0; \overline{2, 1}] = \frac{\sqrt{3}-1}{2}$.

We can define the *tuning operation* $\tau_r := [0, 1] \setminus \mathbb{Q} \mapsto [0, 1]$ as

$$[0; a_1, a_2, \dots] \mapsto [0; S_1 S_0^{a_1-1} S_1 S_0^{a_2-1} \dots]$$

and $\tau_r(0) = \omega$. It is easy to check that the map $x \mapsto \tau_r(x)$ is strictly increasing, and its image is contained in W_r (indeed, $\tau_r(\mathcal{E}) = \mathcal{E} \cap W_r$). Moreover, tuning behaves well with respect to the string action:

$$\tau(S \cdot x) = \tau(S) \cdot \tau(x)$$

where $S = (a_1, \dots, a_n)$ is a finite string and $\tau(S) := S_1 S_0^{a_1-1} \dots S_1 S_0^{a_n-1}$.

The point $\alpha := [0; \overline{S_0}]$ will be called the *root* of the tuning window. A tuning window will be called *maximal* if it is not contained in any other tuning window. An extremal rational number r will be called *primitive* if W_r is a maximal tuning window.

Let us remark that the period-doubling cascades of section 3.1 are generated by iteration of the tuning map $\tau_{1/2}$. For instance, if $g = [0; \overline{1}]$, then $\tau_{1/2}(g) = [0; \overline{2}]$, $\tau_{1/2}^2(g) = [0; \overline{2, 1, 1}]$ and so on, hence $\tau_{1/2}^n(g)$ is the cascade which tends to c_F , and c_F is a fixed point of $\tau_{1/2}$. All other period-doubling cascades are tuned images of this one, namely for any extremal rational number r , the cascade generated by r is just the sequence $\{\tau_r \tau_{1/2}^n(g), n \geq 0\}$.

Let us now observe that the structure of $\mathcal{B}(\omega)$ inside the tuning window is particularly nice:

Lemma 6. *Let $r \in \mathbb{Q} \cap (0, 1)$ be extremal, $W_r = [\omega, \alpha]$. Then*

$$\mathcal{B}(\omega) \cap [\omega, \alpha] = K(\Sigma)$$

where $K(\Sigma)$ is the regular Cantor set on the alphabet $\Sigma = \{S_0, S_1\}$. This means that x belongs to $\mathcal{B}(\omega) \cap [\omega, \alpha]$ if and only if its continued fraction is an infinite concatenation of the strings S_0, S_1 .

Proof. If $x \in \mathcal{B}(\omega) \cap [\omega, \alpha]$, either $x = S_0 \cdot y$ for some y , or $x = S_1 \cdot y$. In the first case, from $x \leq [0; \overline{S_0}]$ and the fact $|S_0|$ is even it follows $y \leq [0; \overline{S_0}]$. In the second case, from $x \geq [0; S_1, \overline{S_0}]$ and the fact $|S_1|$ is odd it also follows that $y \leq [0; \overline{S_0}]$. Moreover, since $x \in \mathcal{B}(\omega)$, $y = G^k(x) \in \mathcal{B}(\omega)$, $y \geq \omega$, hence $y \in \mathcal{B}(\omega) \cap [\omega, \alpha]$ and the inclusion is proved by induction.

Conversely, if $x \in K(\Sigma)$, then $G^k(x) = [0; A, B, \dots]$ with A a suffix of either S_0 or S_1 , and B equal to either S_0 or S_1 . If $A = S_1$, then $G^k(x) \in K(\Sigma)$ and $\min K(\Sigma) = \omega$. If $A \neq S_1$, we claim that in any case $AB \gg S_1$, which implies

$$G^k(x) = [0; A, B, \dots] \geq [0; S_1, \overline{S_0}] = \omega$$

Indeed, if A is suffix of S_0 , then $S_0 = CA$ for some string C . Then either $A = \{1\}$, in which case the claim is trivial since S_1 does not begin with 1, or C is a prefix of B , hence, by weak extremality of S_0 , $AC \geq CA = S_0 \gg S_1$, and, since AC is a prefix of AB , $AB \gg S_1$. The case when A is suffix of S_1 is analogous. \square

We can be even more precise and characterize the action of tuning on \mathcal{E} and $\mathcal{B}(t)$ in the following way:

Proposition 5. *Let $r \in \mathbb{Q} \cap (0, 1)$ be extremal, $W_r = [\omega, \alpha)$. Then, for each $x \in [0, 1] \setminus \mathbb{Q}$*

1.

$$\mathcal{E}(\tau_r(x)) = \mathcal{E}(\alpha) \cup \tau_r(\mathcal{E}(x))$$

2.

$$\mathcal{B}(\tau_r(x)) = \mathcal{B}(\alpha) \cup \bigcup_{S \in \Sigma_\alpha} S \cdot \tau_r(\mathcal{B}(x))$$

where $\Sigma_\alpha := \{SS_0^n : s = [0; S] \text{ is an } \alpha\text{-label}, n \in \mathbb{N}\}$ is a countable set of strings.

Proof. 1. Let $y \in \mathcal{E}(\tau_r(x))$. Either $y \in \mathcal{E} \cap [\alpha, 1] = \mathcal{E}(\alpha)$, or $y \in \mathcal{E} \cap [\tau_r(x), \alpha)$, hence $y \leq [0; \overline{S_1}]$. In the second case, $y \in \mathcal{B}(\omega)$ hence, by lemma 6, the c.f. expansion of y is a concatenation of the strings S_0 and S_1 and begins with S_1 , therefore $y = \tau_r(v)$ for some $v \in [0, 1] \setminus \mathbb{Q}$. Since $G^n(y) \geq y \geq \tau_r(x)$ for all n , then by lemma 7 $G^n(v) \geq v \geq x$, so $v \in \mathcal{E}(x)$. Viceversa, if $v \in \mathcal{E}(x)$, then $G^n(v) \geq v \geq x$ for all n , hence, by lemma 7, $\tau_r(v) \in \mathcal{E}(\tau_r(x))$.

2. Let us assume that

$$(a) \ y \in \mathcal{B}(\tau_r(x)) \quad \text{but} \quad (b) \ y \notin \mathcal{B}(\alpha);$$

from (b) it follows that y is in some α -gap, therefore $y = S \cdot y'$ for some α -label $s = [0; S]$ and $y' < \alpha$. On the other hand, by (a) it follows that $G^{|S|}(y) = y' \in \mathcal{B}(\tau_r(x)) \cap [\omega, \alpha]$, therefore by lemma 6 $y' = [0; Y]$ with Y an infinite concatenation of the words S_0, S_1 and hence $y' = S_0^n \cdot y''$ for some $n \in \mathbb{N}$ and $y'' = \tau_r(v)$ for some $v \in (0, 1) \cap (\mathbb{R} \setminus \mathbb{Q}) \cup \{0\}$. By lemma 7, $v \in \mathcal{B}(x)$, hence $y = SS_0^n \cdot y''$ with $y'' \in \tau_r \mathcal{B}(x)$.

On the other hand, if $y \in \mathcal{B}(\tau_r(x)) \cap [\omega, \alpha]$ and $s = [0; S]$ is an α -label, then $S \cdot y \in \mathcal{B}(\tau_r(x))$: indeed, for all $k < |S|$ we have that $G^k(S \cdot y)$ is between $G^k(s)$ and $G^k(S \cdot \alpha)$, which are both above α . \square

Corollary 3. *Let $r \in \mathbb{Q} \cap (0, 1)$ be extremal, $W_r = [\omega, \alpha)$, and $x \in [0, 1] \setminus \mathbb{Q}$. Then*

$$\text{H.dim } \mathcal{B}(\tau_r(x)) = \max\{\text{H.dim } \mathcal{B}(\alpha), \text{H.dim } \tau_r(\mathcal{B}(x))\} \quad (12)$$

$$\text{H.dim } \mathcal{E}(\tau_r(x)) = \max\{\text{H.dim } \mathcal{E}(\alpha), \text{H.dim } \tau_r(\mathcal{E}(x))\} \quad (13)$$

Remark. We will actually see in section 5 that it is possible to determine precisely which term attains the maximum, i.e.

$$\begin{aligned} \text{H.dim } \mathcal{E}(\tau_r(x)) &= \begin{cases} \text{H.dim } \tau_r \mathcal{E}(x) & \text{if } \alpha > c_F \\ \text{H.dim } \mathcal{E}(\alpha) & \text{if } \alpha < c_F \end{cases} \\ \text{H.dim } \mathcal{B}(\tau_r(x)) &= \begin{cases} \text{H.dim } \tau_r \mathcal{B}(x) & \text{if } \alpha > c_F \\ \text{H.dim } \mathcal{B}(\alpha) & \text{if } \alpha < c_F \end{cases} \end{aligned}$$

where c_F is the fixed point of the tuning operator $\tau_{1/2}$.

Lemma 7. Let $r \in \mathbb{Q} \cap (0, 1)$ be extremal, and $x, y \in [0, 1] \setminus \mathbb{Q}$. Then

$$G^k(x) \geq y \quad \forall k \geq 0$$

if and only if

$$G^k(\tau_r(x)) \geq \tau_r(y) \quad \forall k \geq 0$$

Proof. Since τ_r is increasing, $G^k(x) \geq y$ if and only if $\tau_r(G^k(x)) \geq \tau_r(y)$ if and only if $G^{N_k}(\tau_r(x)) \geq \tau_r(y)$ for $N_k = |S_0|(a_1 + \dots + a_k) + (|S_1| - |S_0|)k$. On the other hand, if h is not of the form N_k , $G^h(\tau_r(x)) = [0; A, B, \dots]$ with A a suffix of either S_0 or S_1 , $A \neq S_1$, and B equal to either S_0 or S_1 . By the same argument as in the proof of lemma 6, in any case $AB \gg S_1$, which implies

$$G^h(\tau_r(x)) = [0; A, B, \dots] \geq [0; S_1, \dots] = \tau_r(y)$$

□

4.1 Scaling and Hölder-continuity at the Feigenbaum point

The tuning operator is useful to understand the scaling law of the Hausdorff dimension at the “Feigenbaum point” c_F . Let $\alpha := [0; \overline{2, 1}]$, $\tau = \tau_{1/2}$, $\alpha_n := \tau^n(\alpha)$ and $g_n := \tau^n(g)$. Since $\tau^n(\mathcal{B}(\alpha)) = \mathcal{B}(\alpha_n) \cap [\alpha_n, g_n]$ we get that

$$\text{H.dim } \tau^n(\mathcal{B}(\alpha)) = \text{H.dim } \mathcal{B}(\alpha_n)$$

We have seen that $\mathcal{B}(\alpha)$ is a regular Cantor set, therefore all its tuned images will be regular Cantor sets as well. Indeed, let us set

$$\begin{cases} Z_0 := (1) \\ Z_1 := (2) \\ Z_{n+1} := Z_n Z_{n-1} Z_{n-1} \text{ if } n \geq 1 \end{cases}$$

it is easy to check by induction that $\tau(Z_n) = Z_{n+1}$, and from this fact it follows that

$$\tau^n(\mathcal{B}(\alpha)) = K(\mathcal{A}_n) \quad \text{with } \mathcal{A}_n := \{Z_n, Z_{n+1}\}$$

If $q_n := q(Z_n)$, then by (3) $q_n^2 \leq q_{n+1} \leq 2q_n^2$ and

$$\begin{aligned} \frac{1}{2q_{n+1}} &\leq |f'_{Z_n}(x)| \leq \frac{1}{q_n^2} \\ \frac{1}{4q_{n+1}^2} &\leq |f'_{Z_{n+1}}(x)| \leq \frac{1}{q_n^4} \end{aligned}$$

Standard Hausdorff dimension estimates ([Fa], Proposition 9.6 and 9.7) then yield

$$\frac{\log \frac{\sqrt{5}+1}{2}}{\log q_{n+1} + \log 2} \leq \text{H.dim } K(\mathcal{A}_n) \leq \frac{\log \frac{\sqrt{5}+1}{2}}{\log q_{n+1} - \log 2} \quad (14)$$

On the other hand, since $\alpha_n = [0; \overline{Z_{n+1}Z_n}]$ has an expansion with a long prefix in common with c_F , we can work out the estimate

$$\frac{1}{96q_{n+1}^5} \leq |c_F - \alpha_n| \leq \frac{2}{q_{n+1}^5} \quad (15)$$

Equations (14) and (15) show the following scaling law for $\mathcal{B}(t)$ at c_F :

$$\lim_{n \rightarrow +\infty} \text{H.dim } \mathcal{B}(\alpha_n) \log \left(\frac{1}{|c_F - \alpha_n|} \right) = 5 \log \frac{\sqrt{5}+1}{2} \quad (16)$$

As a consequence, the map $t \mapsto \text{H.dim } \mathcal{B}(t)$ is not Hölder-continuous at c_F .

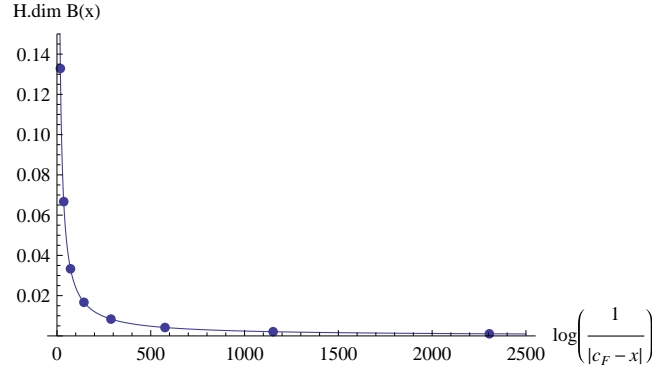


Figure 2: Relation between Hausdorff dimension and distance from c_F . The dots correspond to successive renormalizations α_n , $3 \leq n \leq 10$, while the curve is $xy = 5 \log \frac{\sqrt{5}+1}{2}$.

4.2 Dominant points

In this subsection we will prove proposition 2 stated in the introduction, which will result as a straightforward consequence of proposition 7.

Definition 7. A finite string S of positive integers and even length is called dominant if

$$XY \ll Y$$

for every splitting $S = XY$ where X, Y are finite, nonempty strings. Proposition 2 stated in the introduction is now a straightforward consequence.

Let us remark that every dominant string is extremal. For instance, the strings $(5, 2, 4, 3)$ and $(5, 2, 4, 5)$ are both extremal, but the first is dominant while the second is not.

Definition 8. A point $x \in \mathcal{E}$ is dominant if there exists a dominant word S such that $x = [0; \overline{S}]$.

The importance of dominant points lies in the fact that they approximate all untuned parameters:

Proposition 6. *The set of dominant points is dense in $UT \setminus \{g\}$. More precisely, every parameter in $UT \setminus \{g\}$ is accumulated from the right by dominant parameters.*

The proof of such a fact is quite technical and will be given in section 6.2: assuming the density of dominant points, we can now prove the following:

Proposition 7. *Given any point $x \in UT$ and any $y > x$, then there exists a finite string S such that $S \cdot \mathcal{B}(y) \subseteq \mathcal{E}(x)$.*

Proof. If $x = g$, then $\mathcal{B}(y) = \emptyset$ and the claim is trivial. Otherwise, by proposition 6, x is accumulated from the right by dominant points, hence one can choose $z \in (x, y)$ a dominant point $z = [0; \overline{S}]$ such that S is a prefix of x but not of y . We claim that

$$S \cdot \mathcal{B}(y) \subseteq \mathcal{E}(z)$$

Indeed, let $v \in \mathcal{B}(y)$, $w := S \cdot v$, and $k \geq 0$. Since $v \geq y \geq z$ and S is even, $w = S \cdot v \geq S \cdot z = z$. If $k < |S|$, then there exists a decomposition $S = S' S''$ such that $G^k(w) = S'' \cdot v > S' \cdot v = w$ by dominance. On the other hand, if $k \geq |S|$ then $G^k(w) = G^h(v) \geq y$ for $h := k - |S|$ and $y > S \cdot v = w$ because S is not a prefix of y , hence $G^k(w) > w$.

As a consequence, the set $S \cdot \mathcal{B}(y) \subseteq \mathcal{E}(z) \subseteq \mathcal{E}(x)$ is a Lipschitz image of $\mathcal{B}(y)$ inside $\mathcal{E}(x)$. \square

5 Fractal dimension

We are now ready to prove theorem 2, namely that, for each $t \in [0, 1]$,

$$\text{H.dim } \mathcal{B}(t) = \text{H.dim } \mathcal{E}(t)$$

where $\mathcal{E}(t) := \mathcal{E} \cap [t, 1]$. Let us recall that by definition $\mathcal{E}(x) \subseteq \mathcal{B}(x)$, hence one inequality is trivial. Moreover, since both functions $x \mapsto \mathcal{B}(x)$ and $x \mapsto \mathcal{E}(x)$ are locally constant around any x which does not belong to \mathcal{E} , it is enough to prove that

$$\text{H.dim } \mathcal{B}(x) = \text{H.dim } \mathcal{E}(x) \quad \text{for all } x \in \mathcal{E} \tag{17}$$

Let $\tau = \tau_{\frac{1}{2}}$ be the tuning operation corresponding to $\frac{1}{2}$. Note that $W_{\frac{1}{2}} = [g^2, g]$ where g is the golden mean, and τ has a unique fixed point $c_F \in [g^2, g]$. By the estimates of section 4.1, $\text{H.dim } \mathcal{B}(c_F) = 0$, and (17) holds for $x = c_F$.

Given any $x \in \mathcal{E} \setminus \{c_F\}$, then there exists $n \geq 0$ and $y \in \mathcal{E} \setminus [g^2, g]$ such that $x = \tau^n(y)$. Now, by corollary 3 and the fact $\mathcal{E}(g) = \mathcal{B}(g) = \{g\}$ is a single point,

$$\text{H.dim } \mathcal{B}(\tau^n(y)) = \text{H.dim } \tau^n \mathcal{B}(y)$$

$$\text{H.dim } \mathcal{E}(\tau^n(y)) = \text{H.dim } \tau^n \mathcal{E}(y)$$

We have two cases:

1. If $y \in UT$, then by proposition 2 for each $z > y$ there exists a finite string S such that $S \cdot \mathcal{B}(z) \subseteq \mathcal{E}(y)$, hence by applying τ^n to both sides

$$\tau^n(S) \cdot \tau^n(\mathcal{B}(z)) = \tau^n(S \cdot \mathcal{B}(z)) \subseteq \tau^n(\mathcal{E}(y))$$

hence $\tau^n(\mathcal{E}(y))$ contains a Lipschitz image of $\tau^n(\mathcal{B}(z))$ and

$$\sup_{z > y} \text{H.dim } \mathcal{B}(\tau^n(z)) = \sup_{z > y} \text{H.dim } \tau^n \mathcal{B}(z) \leq \text{H.dim } \tau^n \mathcal{E}(y) = \text{H.dim } \mathcal{E}(\tau^n y)$$

Moreover, by continuity of Hausdorff dimension and of τ ,

$$\sup_{z > y} \text{H.dim } \mathcal{B}(\tau^n z) = \text{H.dim } \mathcal{B}(\tau^n y)$$

hence $\text{H.dim } \mathcal{B}(x) \leq \text{H.dim } \mathcal{E}(x)$ and (17) follows.

2. If $y \notin UT$, then $y = \tau_r(z)$ for some extremal rational $r < g^2$ and some $z \in [0, 1] \setminus \mathbb{Q}$. Now, applying the tuning operation to both sides of formula 1. from proposition 5

$$\text{H.dim } \tau^n \mathcal{E}(\tau_r(z)) = \max\{\text{H.dim } \tau^n \tau_r \mathcal{E}(z), \text{H.dim } \tau^n \mathcal{E}(\alpha_r)\}$$

where α_r is the root of the tuning window relative to τ_r , and the same formula with \mathcal{E} replaced by \mathcal{B} holds by formula 2. of prop. 5. Now, since $\alpha_r < g^2$

$$\tau^n \mathcal{E}(\alpha_r) \supseteq \tau^n \mathcal{E}(g^2) = \tau^n \mathcal{E}(\tau(0)) \supseteq \tau^{n+1} \mathcal{E}$$

while

$$\tau^n \tau_r \mathcal{E}(z) \subseteq \tau^n \tau_r \mathcal{E}$$

and since the map $\tau^n \tau_r \tau^{-(n+1)}$ restricted to numbers of bounded type is Lipschitz (lemma 8), then $\text{H.dim } \tau^n \tau_r \mathcal{E} \leq \text{H.dim } \tau^{n+1} \mathcal{E}$ and

$$\text{H.dim } \tau^n \tau_r \mathcal{E}(z) \leq \text{H.dim } \tau^n \tau_r \mathcal{E} \leq \text{H.dim } \tau^{n+1} \mathcal{E} \leq \text{H.dim } \tau^n \mathcal{E}(\alpha_r)$$

hence $\text{H.dim } \tau^n \mathcal{E}(\tau_r z) = \text{H.dim } \tau^n \mathcal{E}(\alpha_r)$ and similarly $\text{H.dim } \tau^n \mathcal{B}(\tau_r z) = \text{H.dim } \tau^n \mathcal{B}(\alpha_r)$. Since $\alpha_r \in UT$, by case 1. $\text{H.dim } \mathcal{E}(\tau^n \alpha_r) = \text{H.dim } \mathcal{B}(\tau^n \alpha_r)$, hence

$$\begin{aligned} \text{H.dim } \mathcal{E}(\tau^n \tau_r z) &= \text{H.dim } \tau^n \mathcal{E}(\tau_r z) = \text{H.dim } \tau^n \mathcal{E}(\alpha_r) = \text{H.dim } \mathcal{E}(\tau^n \alpha_r) = \\ &= \text{H.dim } \mathcal{B}(\tau^n \alpha_r) = \text{H.dim } \tau^n \mathcal{B}(\alpha_r) = \text{H.dim } \tau^n \mathcal{B}(\tau_r z) = \text{H.dim } \mathcal{B}(\tau^n \tau_r z) \end{aligned}$$

and the theorem follows. □

Lemma 8. For each $n \geq 0$, $t > 0$ and $r \in \mathbb{Q} \cap (0, 1)$ extremal, the map $\tau^n \tau_r \tau^{-(n+1)} : \tau^{n+1} \mathcal{B}(t) \rightarrow \tau^n \tau_r \mathcal{B}(t)$ is Lipschitz continuous.

Proof. Let us define the partial order \leq_* on the set of finite strings of positive integers: given two strings $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_m)$, $A \leq_* B$ if there exists an increasing function $K : \{1, \dots, n\} \mapsto \{1, \dots, m\}$ such that for all $1 \leq i \leq n$, $a_i \leq b_{K(i)}$. Notice that \leq_* behaves well under concatenation ($A \leq_* B, C \leq_* D \Rightarrow AC \leq_* BD$) and tuning (i.e. $A \leq_* B \Rightarrow \tau_r(A) \leq_* \tau_r(B)$)

for any r). Moreover, if $A \leq_\star B$, then $q(A) \leq q(B)$. Let us now consider $x, y \in \mathcal{B}(t)$. There exists a string S such that $x = S \cdot x', y = S \cdot y'$ with x', y' without any common digit at the beginning of their continued fraction expansion. Notice that since $x', y' \in \mathcal{B}(t)$, there exists $C > 0$ (which depends on t and n) such that $d(\tau^{n+1}(x'), \tau^{n+1}(y')) \geq C$. Let us denote by Σ_1 and Σ_2 the two finite strings $\Sigma_1 := \tau^n \tau_r(S)$, $\Sigma_2 := \tau^{n+1}(S)$, and S_1, S_0 be the even and odd string representing r . Let us notice that by eq. (4)

$$d(\tau^n \tau_r(x), \tau^n \tau_r(y)) \leq \sup |f'_{\Sigma_1}| d(\tau^n \tau_r(x'), \tau^n \tau_r(y')) \leq \frac{1}{q(\Sigma_1)^2}$$

$$d(\tau^{n+1}(x), \tau^{n+1}(y)) \geq \inf |f'_{\Sigma_2}| d(\tau^{n+1}(x'), \tau^{n+1}(y')) \geq \frac{C}{4q(\Sigma_2)^2}$$

hence

$$d(\tau^n \tau_r(x), \tau^n \tau_r(y)) \leq \frac{4q(\Sigma_2)^2}{q(\Sigma_1)^2 C} d(\tau^{n+1}(x), \tau^{n+1}(y))$$

Now, since $(1, 1) \leq_\star S_0$ and $(2) \leq_\star S_1$, then by concatenation $\tau(S) \leq_\star \tau_r(S)$, and by tuning $\Sigma_2 = \tau^{n+1}(S) \leq_\star \tau^n \tau_r(S) = \Sigma_1$, hence $q(\Sigma_2) \leq q(\Sigma_1)$ and the claim follows. \square

5.1 A problem about Sturmian sequences

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, a *Sturmian sequence* of slope α is a binary sequence of the type

$$S_{\alpha, \beta} = \lfloor \alpha(n+1) + \beta \rfloor - \lfloor \alpha n + \beta \rfloor \quad \text{or} \quad S_{\alpha, \beta} = \lceil \alpha(n+1) + \beta \rceil - \lceil \alpha n + \beta \rceil \quad (18)$$

where β is some other real value. Sturmian sequences have also a geometrical interpretation: they can be viewed as cutting sequences of half-lines on the plane with respect to the integral lattice \mathbb{Z}^2 .

Given a sequence X (finite or infinite) and a positive integer m , the set of m -factors of X is the set of substrings of X of length m :

$$\mathcal{F}_m(X) := \{S = (x_{n+1}, \dots, x_{n+m}) : 0 \leq n < |X| - m + 1\}$$

The *recurrence function* of a binary sequence $X \in \{0, 1\}^{\mathbb{N}}$ is the function $R_X : \mathbb{N} \rightarrow \mathbb{N} \cup \{+\infty\}$ defined by²

$$R_X(n) := \inf\{m \in \mathbb{N} : \forall S \in \mathcal{F}_m(X), \mathcal{F}_n(S) = \mathcal{F}_n(X)\}$$

while the *recurrence quotient* of X is the maximal linear growth rate of $R_X(n)$:

$$R_X := \limsup_{n \rightarrow +\infty} \frac{R_X(n)}{n}$$

It is well known that the recurrence quotient of a Sturmian sequence $S_{\alpha, \beta}$ depends only on the continued fraction expansion of its slope $\alpha = [a_0; a_1, a_2, \dots]$. In fact, the following formula holds ([Ca], corollary 1)

$$R_{S_{\alpha, \beta}} = \rho(\alpha) := 2 + \limsup_{k \rightarrow +\infty} [a_k; a_{k-1}, a_{k-2}, \dots, a_1]$$

²We follow the usual convention $\inf \emptyset = +\infty$.

So, if $\limsup a_k = N$, then $\rho(\alpha) \in (N + 2, N + 3)$; if otherwise α has unbounded partial quotients, then $\rho(\alpha) = +\infty$. The *recurrence spectrum of Sturmian sequences* is defined by

$$\mathcal{R} := \{\rho(\alpha), \alpha \in \mathbb{R} \setminus \mathbb{Q}\};$$

the main result of Cassaigne is a characterization of \mathcal{R} that, in terms of \mathcal{E} , can be written as follows (c.f. [Ca], theorem 1):

$$\mathcal{R} = \{2 + \frac{1}{x} : x \in \mathcal{E}\} \quad (19)$$

Cassaigne posed the question of determining the Hausdorff dimension of the sections $\mathcal{R} \cap [N + 2, N + 3]$ for each positive integer N : the answer to this question is a corollary of theorem 2, in fact:

$$\text{H.dim } \mathcal{R} \cap [N+2, N+3] = \text{H.dim } \mathcal{E} \cap \left[\frac{1}{N+1}, \frac{1}{N} \right] = \text{H.dim } \mathcal{E} \left(\frac{1}{N+1} \right) = \text{H.dim } \mathcal{B}_N.$$

6 Technical proofs

6.1 Continuity of Hausdorff dimension

The goal of this subsection is to prove theorem 1, namely that the function $t \mapsto \text{H.dim } \mathcal{B}(t)$ is continuous.

The proof uses the same technique used in [Mo] to prove continuity of the Hausdorff dimension of sections of the Markoff spectrum.

Let us recall that, if $S = (a_1, \dots, a_n)$ is a finite string of positive integers, the *cylinder set* defined by S is the set

$$I(S) := \{x = S \cdot y, y \in [0, 1]\}$$

which is a closed interval with endpoints $[0; a_1, \dots, a_n]$ and $[0; a_1, \dots, a_n + 1]$.

Given $r > 0$, one can cover the compact set $\mathcal{B}(t)$ with cylinder sets for the Gauss map in such a way that they all have roughly the same size: more precisely, we can consider the set

$$C(t, r) := \{S = (a_1, \dots, a_n) \mid I(S) \cap \mathcal{B}(t) \neq \emptyset, |I(S)| \leq e^{-r} \text{ and } |I(\check{S})| > e^{-r}\}$$

where $\check{S} = (a_1, \dots, a_{n-1})$. Let us remark that the cylinders associated to any two distinct elements of $C(t, r)$ have disjoint interiors, and their union is a finite cover of $\mathcal{B}(t)$. We will denote by $N(t, r)$ the cardinality of $C(t, r)$.

By equation (5), one also gets the lower bound

$$|I(S)| \geq c_t |I(\check{S})| \geq c_t e^{-r} \quad c_t = \frac{t^2}{2(1+t)^2}$$

Let us now define the function

$$D(t) := \inf_{r \in \mathbb{N}} \frac{\log N(t, r)}{r}$$

which counts the growth rate of the number of cylinders of size $\cong e^{-r}$ needed to cover $\mathcal{B}(t)$; subadditivity of $\log N(t, r)$ yields

$$D(t) = \lim_{r \rightarrow \infty} \frac{\log N(t, r)}{r}$$

Since the cylinders corresponding to $C(t, r)$ form a cover of $\mathcal{B}(t)$, it is immediate to see that

$$\text{H.dim } \mathcal{B}(t) \leq D(t) \quad (20)$$

The main idea in order to bound the Hausdorff dimension from below is to approximate $\mathcal{B}(t)$ with *regular Cantor sets* (recall definition 2). The following technical lemma is an adaptation of lemma 1 in [Mo]:

Lemma 9. *Given $t \in [0, 1]$ and $\eta \in (0, 1)$, there exists a regular Cantor set $K(\mathcal{A})$ and some $\delta > 0$ such that*

$$\text{H.dim } K(\mathcal{A}) \geq (1 - \eta)D(t) \quad (21)$$

and

$$K(\mathcal{A}) \subseteq \mathcal{B}(t + \delta) \quad (22)$$

It is easy to see how the theorem follows from the lemma: the latter implies

$$D(t) \leq \sup_{t' > t} \text{H.dim } \mathcal{B}(t')$$

hence from (20)

$$\text{H.dim } \mathcal{B}(t) \leq D(t) \leq \sup_{t' > t} \text{H.dim } \mathcal{B}(t) \leq \text{H.dim } \mathcal{B}(t)$$

so $\text{H.dim } \mathcal{B}(t) = D(t)$, and such function is right-continuous. Left-continuity of $D(t)$ is easy:

$$\inf_{t' < t} D(t') = \inf_{t' < t} \inf_r \frac{\log N(t', r)}{r} = \inf_r \frac{1}{r} \inf_{t' < t} \log N(t', r) = \inf_r \frac{1}{r} \log N(t, r) = D(t)$$

where we used the compactness of $\mathcal{B}(t)$ and cylinder sets to conclude that

$$\bigcap_{t' < t} C(t', r) = C(t, r)$$

and this proves the theorem. \square

Proof of lemma 9. We can choose r sufficiently large such that

$$\left| \frac{\log N(t, r)}{r} - D(t) \right| < \frac{\eta}{80} D(t)$$

We will denote $N := N(t, r)$, $\mathcal{A}_0 := C(t, r)$ and $l := \max\{|W_i| : W_i \in \mathcal{A}_0\}$. With such choices, Moreira proves that there exists an alphabet \mathcal{A} such that:

1. Each word $W \in \mathcal{A}$ is the concatenation of exactly s words of \mathcal{A}_0

2. For each word $W = W_1 \dots W_s \in \mathcal{A}$, $W_i \in \mathcal{A}_0$ there exist $W'_1, W''_1, W'_s, W''_s \in \mathcal{A}_0$ such that

$$W'_1 \ll W_1 \ll W''_1 \quad W'_s \ll W_s \ll W''_s$$

and

$$I(W_1 \dots W_{s-1} W'_s) \cap \mathcal{B}(t) \neq \emptyset \quad I(W_2 \dots W_{s-1} W_s W'_1) \cap \mathcal{B}(t) \neq \emptyset$$

3. The total number of words is $|\mathcal{A}| \geq N^{(1-\frac{\eta}{4})s}$

First of all, let us check equation (21): since any $W \in \mathcal{A}$ is of the form $W = W_1 \dots W_s$ where $|I(W_i)| \geq c_t e^{-r}$, by equations (4) and (5) we have the following lower bound for the contraction rate:

$$\inf_{x \in [0,1]} |f'_W(x)| \geq \prod_{i=1}^s \inf_{x \in [0,1]} |f'_{W_i}| \geq \frac{1}{2^s} \prod_{i=1}^s \frac{1}{q(W_i)^2} \geq \frac{1}{2^s} \prod_{i=1}^s |I(W_i)| \geq (2e^r c_t^{-1})^{-s}$$

and since there are at least $N^{(1-\frac{\eta}{4})s}$ cylinders, then by eq. (7)

$$\text{H.dim } K(\mathcal{A}) \geq \frac{s(1-\frac{\eta}{4}) \log N}{s \log(2e^r c_t^{-1})} \geq (1-\eta)D(t)$$

where in the last inequality we used that $\frac{\log N}{r} \geq (1-\eta/80)D(t)$ and r is large enough.

Let us now verify (22). First of all, let us notice that there exists trivially some $F := \mathcal{B}(\frac{1}{N})$ which contains both $K(\mathcal{A})$ and $\mathcal{B}(t)$. Let $x = [0; W_1 \dots W_{s-1} W_s W_1 \dots] \in K(\mathcal{A})$ and $k \geq 0$. Then $G^k(x)$ is either of the form

$$G^k(x) = SW_s \cdot y, \quad |S| \leq |W_1| + |W_2| + \dots + |W_{s-1}| \leq (s-1)l \quad y \in F$$

or

$$G^k(x) = SW_1 \cdot y, \quad |S| \leq |\beta_s| \leq l \quad y \in F$$

Let us consider the first case (the second is analogous). By previous point 2. one can choose $W^* \in \{W'_s, W''_s\}$ such that

$$SW_s \gg SW^*$$

Then, for any $z_1, z_2 \in F$

$$d(SW_s \cdot z_1, SW^* \cdot z_2) \geq \lambda^{|S|} d(I(W_s) \cap F, I(W^*) \cap F)$$

where $\lambda = \frac{1}{\sup_F |G'|}$ is the maximum contraction factor of the inverse branches of the Gauss map. Let us moreover notice that

$$d := \inf_{\substack{W_i, W_j \in \mathcal{A}_0 \\ W_i \neq W_j}} d(I(W_i) \cap F, I(W_j) \cap F) > 0$$

since the cylinders $I(W_i)$ have disjoint interiors and F is a compact set which contains no rational number. Now, by previous point 2. one can find $z \in \mathcal{B}(t)$ such that

$$z = W_1 \dots W_{s-1} W^* \cdot w, \quad w \in F$$

hence $SW^* \cdot w \geq t$ and

$$G^k(x) = SW_s \cdot y \geq \lambda^{|S|} d + SW^* \cdot w \geq \lambda^{(s-1)l} d + t$$

hence the claim is proved with $\delta := \lambda^{(s-1)l} d$. \square

6.2 Density of dominant points

In order to prove proposition 6, we will need the following definitions: given a string S , the set of its *prefixes-suffixes* is

$$\begin{aligned} PS(S) &:= \{Y : Y \text{ is both a prefix and a suffix of } S\} = \\ &= \{Y : Y \neq \emptyset, \exists X, Z \text{ s.t. } S = XY = YZ\} \end{aligned}$$

Moreover, we have the set of *residual suffixes*

$$RS(S) := \{Z : S = YZ, Y \in PS(S)\}$$

Proof of proposition 6. By density of the roots of the maximal tuning windows, it is enough to prove that every α which is root of a maximal tuning window, $\alpha \neq g$, can be approximated from the right by dominant points. Hence we can assume $\alpha = [0; \overline{S}]$, S an extremal word of even length, and 1 is not a prefix of S . If S is dominant, a sequence of approximating dominant points is given by $[0; S^n, 11]$, $n \geq 1$. The rest of the proof is by induction on $|S|$. If $|S| = 2$, then S itself is dominant and we are in the previous case. If $|S| > 2$, either S is dominant and we are done, or $PS(S) \neq \emptyset$ and also $RS(S) \neq \emptyset$. Let us choose $Z_* \in RS(S)$ such that

$$[0; \overline{Z_*}] := \min\{[0; \overline{Z}] : Z \in RS(S)\}$$

and $Y_* \in PS(S)$ such that $S = Y_*Z_*$. Let $\alpha(Y_*)$ be the root of the maximal tuning window $[0; Y_*]$ belongs to. Then by lemma 13, $[0; \overline{Z_*}] > \alpha(Y_*)$, and by minimality

$$\alpha(Y_*) < [0; \overline{Z}] \quad \forall Z \in RS(S)$$

Now, $\alpha(Y_*) = [0; \overline{P}]$ with either $P = \{1\}$, which is dominant, or $|P|$ even. Moreover,

$$|P| \leq |Y_*| + 1 \leq |S|$$

and actually $|Y_*| + 1 < |S|$ because otherwise the first digit of Y_* would appear twice at the beginning of S , contradicting the fact that S is extremal. Hence $|P| < |S|$ and by induction there exists $\gamma = [0; \overline{T}]$ such that T is dominant,

$$\alpha(Y_*) < [0; \overline{T}] < [0; \overline{Z}] \quad \forall Z \in RS(S)$$

and γ can also be chosen close enough to $\alpha(Y_*)$ so that P is prefix of T , which implies

$$S << T$$

By lemma 11, $S^n T^m$ is a dominant word for m large enough, of even length if m is even, and arbitrarily close to $[0; \overline{S}]$ as n tends to infinity. \square

Lemma 10. *If S is an extremal word and $Y \in PS(S)$, then Y is an extremal word of odd length.*

Proof. Suppose $S = XY = YZ$. Then by extremality $XY < YX$, hence $XXY < YXY$ and, by substituting YZ for XY , $YZY < YYZ$. If $|Y|$ were even, it would follow that $ZY < YZ$, which contradicts the extremality of $S = YZ < ZY$. Hence $|Y|$ is odd. Suppose now $Y = AB$, with A and B non-empty strings. Then $S = XAB < BXA$. By considering the first $k := |Y|$

characters on both sides of this equation, $Y = AB = S_1^k \leq BXA_1^k = BA$. If $Y = AB = BA$, then $Y = P^k$ for some string P , hence by the string lemma (eq. (6) of section 2.1) $P^2ZP^{k-2} < P^kZ = S$, which contradicts the extremality of S , hence $AB < BA$ and Y is extremal. \square

Lemma 11. *Let S be an extremal word of even length, and T be a dominant word. Suppose moreover that*

1. $S << T$
2. $\bar{T} < \bar{Z} \quad \forall Z \in RS(S)$

for any $n \geq 1$ and for m sufficiently large, S^nT^m is a dominant word.

Proof. 1. From 1.,

$$\begin{aligned} S^nT^m &<< T^a, \quad a \geq 1 \\ S^nT^m &<< S^bT^m, \quad b < n \end{aligned}$$

2. If $S = xy$, $xy << yx$ by extremality, hence

$$S^nT^m << yS^bT^m \quad \forall b \geq 1$$

3. Since T is dominant, $T << u$ whenever $T = tu$, thus

$$S^nT^m << T << u$$

4. One is left to prove that $S^nT^m << yT^m$ whenever $S = xy$. If $y \notin PS(S)$, then $xy << y$ and the proof is complete. Otherwise, $S = xy = yz$, $|y| \equiv 1 \pmod 2$ by lemma 10. Moreover, since $yz < zy$, $[0; zS^{n-1}] > [0; \bar{z}]$, hence 2. implies $[0; \bar{T}] < [0; z, S^{n-1}]$, and by lemma 12 $[0; zS^{n-1}\bar{T}] > [0; \bar{T}]$, hence for m large enough $zS^{n-1}T^m >> T^m$ and then

$$S^nT^m << yT^m$$

\square

Lemma 12. *Let Y, Z be finite strings of positive integers such that $[0; \bar{Y}] < [0; \bar{Z}]$. Then*

$$[0; Z, \bar{Y}] > [0; \bar{Y}]$$

Proof. By the string lemma (6) of section 2.1, for any $k \geq 0$

$$[0; \bar{Y}^k] < [0; \bar{Z}] \Rightarrow Y^kZ < ZY^k$$

hence, by taking the limit as $k \rightarrow \infty$, $[0; Z, \bar{Y}] \geq [0; \bar{Y}]$. Equality cannot hold because otherwise Y and Z have to be multiple of the same string, which contradicts the strict inequality $[0; \bar{Y}] < [0; \bar{Z}]$. \square

Lemma 13. *Let S be an extremal word of even length such that $\gamma = [0; \bar{S}] \in UT$, and let $Y \in PS(S)$, $S = YZ$. Let $\alpha(Y)$ be the root of the maximal tuning window W_r which contains $[0; Y]$. Then*

$$[0; \bar{Z}] > \alpha(Y)$$

Proof. Let $r = [0; S_0] = [0; S_1]$ be the expansions of r in c.f. of even and odd length, respectively. Since $[0; Y]$ lies in W_r and $|Y|$ is odd, then Y is a concatenation of the strings S_0 and S_1 ; we claim

$$\beta := [0; \overline{ZY}] > \alpha(Y) = [0; \overline{S_0}]$$

Indeed, suppose $\beta \leq \alpha(Y)$; since the c.f. expansion of $Y \cdot \alpha(Y)$ is a concatenation of strings S_0 and S_1 , then $Y \cdot \alpha(Y) \geq [0; S_1, \overline{S_0}] = \omega(Y)$ where $\omega(Y)$ is the lower endpoint of the tuning window W_r . Hence, from $\beta \leq \alpha(Y)$, then $\gamma = Y \cdot \beta \geq Y \cdot \alpha(Y) \geq \omega(Y)$. On the other hand, $\gamma = [0; \overline{S}] < [0; \overline{Y}] < \alpha(Y)$ by the string lemma (eq. (6) of section 2.1), so $\gamma \in W_r$, which contradicts the hypothesis that γ is untuned.

Now, if $[0; \overline{Z}] \leq \alpha(Y)$, then Z has to be prefix of $\overline{S_0}$, hence $Z = S_0^k V$ with V prefix of S_0 , $V \neq \emptyset$ since $|Z|$ is odd. If $S_0 \neq \{1, 1\}$, then S_0 is extremal and by the string lemma $[0; \overline{Z}] = [0; \overline{S_0^k V}] > [0; \overline{S_0}]$, contradiction. In the case $S_0 = \{1, 1\}$ one also has to exclude $Z = \{1\}$. \square

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